Differential Forms and the Wodzicki Residue for Manifolds with Boundary *

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Abstract In [3], Connes found a conformal invariant using Wodzicki's 1-density and computed it in the case of 4-dimensional manifold without boundary. In [14], Ugalde generalized the Connes' result to n-dimensional manifold without boundary. In this paper, we generalize the results of [3] and [14] to the case of manifolds with boundary.
Subj. Class.: Noncommutative global analysis; Noncommutative differential geometry.
MSC: 58G20; 53A30; 46L87
Keywords: Boutet de Monvel's algebra; Wodzicki's residue for manifolds with boundary; Subconformal manifolds; Subconformal invariant.

1 Introduction

In 1984, Wodzicki discovered a trace on the algebra $\Psi_{\rm cl}(X)$ of all classical pseudodifferential operators on a closed compact manifold X in [15], which vanishes if the order of the operator is less than $-{\rm dim}X$. It turns out to be the unique trace on this algebra up to rescaling.

Wodzicki' residue has been applied to many branches of mathematics. Especially, it plays a prominent role in noncommutative geometry. In [4], Connes proved that Wodzicki's residue coincided with Dixmier's trace on pseudodifferential operators of order $-{\rm dim}X$. Wodzicki's residue also had been used to derive an action for gravity in the framework of noncommutative geometry in [6],[9],[10].

In [3], for an even dimensional compact oriented conformal manifold X without boundary, Connes constructs a canonical Fredholm module (H, F). Here H is the Hilbert space of square integrable forms of middle dimension: $H = L^2(X, \wedge_c^l T^*X)$ with $l = \frac{1}{2}{\rm dim}X$ and $F = 2P - 1$ where P is the orthogonal projection on the image of d. By Hodge decomposition theorem, we observe that F preserves the finite

*partially supported by MOEC and the 973 project.

dimensional space of harmonic forms H^l, and F restricted to $H \ominus H^l$ is given by

$$F = \frac{d\delta - \delta d}{d\delta + \delta d}.$$

Using the equality :

$$\mathrm{Wres}(f_0[F, f_1][F, f_2]) = \int_X f_0 \Omega_n(f_1, f_2), \tag{1.1}$$

where $f_0,\ f_1,\ f_2 \in C^\infty(X)$, Connes defined an n-form $\Omega_n(f_1, f_2)$ which is uniquely determined, symmetric in f_1 and f_2, and conformally invariant. In particular, in the 4-dimensional case, this differential form was explicitly computed in [3] by the conformal deformation way.

In [13], Ugalde presented the computations in the six dimensional case for a whole family of differential forms related to $\Omega_n(f_1, f_2)$. In [14], he gave an explicit expression of $\Omega_n(f_1, f_2)$ in the flat case and indicated the way of computation in the general case.

The purpose of this paper is to generalize these results to the case of manifolds with boundary.

To do so, we find first that Wodzicki's residue in (1.1) should be replaced by Wodzicki' residue for manifolds with boundary. For a detailed introduction to the residue for manifolds with boundary see [5], where Fedosov etc. defined a residue on Boutet de Monvel's algebra and proved that it is a unique continuous trace. For a good summary also see [11]. In addition, Grubb and Schrohe got this residue through asymptotic expansions in [8]. Subsequently we will use operator π^+F in Boutet de Monvel's algebra instead of F in (1.1) (π^+F will be introduced in Section 2.1).

Secondly, we will use the form pair $(\Omega_{n,\pi^+S,X}(f_1, f_2), \Omega_{n-1,\pi^+S,Y}(f_1, f_2))$ instead of $\Omega_n(f_1, f_2)$ where $Y = \partial X$ and (1.1) turns into:

$$\begin{aligned}&\widetilde{\mathrm{Wres}}(\pi^+ f_0[\pi^+F, \pi^+ f_1][\pi^+F, \pi^+ f_2])\\ &\qquad = \int_X f_0 \Omega_{n,\pi^+F,X}(f_1, f_2) + \int_Y f_0|_Y \Omega_{n-1,\pi^+F,Y}(f_1, f_2),\end{aligned} \tag{1.2}$$

where $f_0, f_1, f_2 \in C^\infty(X)$; $f_0|_Y$ denotes that the restriction of f_0 on Y. Here f_0 is assumed to be independent of x_n near the boundary, where $x' = (x_1, ..., x_{n-1})$ are coordinates on ∂X and x_n is the normal coordinate (In what follows, x_n always denotes the normal coordinate.).

In Section 2, we briefly recall Boutet de Monvel's calculus and Wodzicki's residue for manifolds with boundary.

In Section 3, for a pseudodifferential operator S of order 0 with the transmission property acting on sections of a vector bundle E over $\widetilde{X} = X \cup_Y X$, we consider $\widetilde{\mathrm{Wres}}(\pi^+ f_0[\pi^+S, \pi^+ f_1][\pi^+S, \pi^+ f_2])$ where $\pi^+S : C^\infty(X, E|_X) \to C^\infty(X, E|_X)$ is defined in Section 2.1. We also show that

$$\Omega_{n,\pi^+S,X}(f_1, f_2) = \Omega_{n,S,\tilde{X}}(\bar{f}_1, \bar{f}_2)|_X\ ; \tag{1.3}$$

$$\widetilde{\mathrm{Wres}}(\pi^+f_0[\pi^+S,\pi^+f_1][\pi^+S,\pi^+f_2])$$
$$=\int_X f_0\Omega_{n,\pi^+S,X}(f_1,f_2)+\int_Y f_0|_Y\Omega_{n-1,\pi^+S,Y}(f_1,f_2). \tag{1.4}$$

determine a unique form pair $(\Omega_{n,\pi^+S,X}(f_1,f_2),\Omega_{n-1,\pi^+S,Y}(f_1,f_2))$ which is symmetric in f_1 and f_2, where $\bar{f}_1,\bar{f}_2$ are extensions on $\tilde{X}$ of f_1,f_2 and f_0 is independent of x_n near the boundary and $\Omega_{n,S,\tilde{X}}(\bar{f}_1,\bar{f}_2)=\Omega_n(\bar{f}_1,\bar{f}_2)$ is defined in [3]. Moreover,$\widetilde{\mathrm{Wres}}(\pi^+f_0[\pi^+S,\pi^+f_1][\pi^+S,\pi^+f_2])$ is a Hochschild 2-cocycle (see Section 3) over $C^\infty(X)$.

In Section 4, for a Riemannian manifold (X,g) which has the product metric near the boundary, $(\tilde{X},\tilde{g})$ is the associated double Riemannian manifold. When $\dim X$ is even, then $\Omega_{n-1,\pi^+F,Y}(f_1,f_2)=0$ and we get the formula:

$$\widetilde{\mathrm{Wres}}(\pi^+f_0[\pi^+F,\pi^+f_1][\pi^+F,\pi^+f_2])=\int_X f_0\Omega_{n,\pi^+F,X}(f_1,f_2). \tag{1.5}$$

So we define subconformal manifolds and $\Omega_{n,\pi^+F,X}(f_1,f_2)$ is a obvious subconformal invariant. When $\dim X$ is odd and $(S,E)=(F,H)$, where $H=L^2(X,\wedge_c^{\frac{n+1}{2}}T^\star X)$ and F is defined as before, then $\Omega_{n,\pi^+F,X}(f_1,f_2)=0$. So we get:

$$\widetilde{\mathrm{Wres}}(\pi^+f_0[\pi^+F,\pi^+f_1][\pi^+F,\pi^+f_2])=\int_Y f_0|_Y\Omega_{n-1,\pi^+F,Y}(f_1,f_2). \tag{1.6}$$

Subsequently, in Sections 5,6, we compute the expression of $\Omega_{n-1,\pi^+F,Y}(f_1,f_2)$ and get its explicit expression for flat manifolds in the x_n-independent and the x_n-dependent cases. In Section 7, when $n=3$, using the normal coordinate way we prove the formula:

$$\Omega_{2,\pi^+F,Y}(f_1,f_2)=\frac{3}{8}\pi\Omega_2(f_1|Y,f_2|Y)-6\pi^2\partial_{x_n}f_1|_{x_n=0}\partial_{x_n}f_2|_{x_n=0}\mathrm{Vol}_Y. \tag{1.7}$$

So
$$\Omega_{2,\pi^+F,Y}(f_1,f_2)+6\pi^2\partial_{x_n}f_1|_{x_n=0}\partial_{x_n}f_2|_{x_n=0}\mathrm{Vol}_Y \tag{1.8}$$
may be considered as a conformal invariant of (X,g). The above results generalize [3] and [14] to the case of manifolds with boundary.

For the rest of this paper, We will briefly use $(\Omega_n(f_1,f_2),\Omega_{n-1}(f_1,f_2))$ instead of $(\Omega_{n,\pi^+S,X}(f_1,f_2),\Omega_{n-1,\pi^+S,Y}(f_1,f_2))$ in this section. We refer $\Omega_n(f_1,f_2)$ in this paper (in [3]) if f_1 and f_2 are functions on manifolds with (without) boundary.

2 Boutet de Monvel's Calculus and Residue for Manifolds with Boundary

In this section, we recall some basic facts about Boutet de Monvel's calculus which will be used in the following. For more details, see [1], [7], [11] and [12].

2.1 Boutet de Monvel's Algebra

Let

$$F : L^2(\mathbf{R}_t) \to L^2(\mathbf{R}_v);\ F(u)(v) = \int e^{-ivt} u(t)dt$$

denote the Fourier transformation and $\Phi(\overline{\mathbf{R}^+}) = r^+\Phi(\mathbf{R})$ (similarly define $\Phi(\overline{\mathbf{R}^-})$), where $\Phi(\mathbf{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbf{R}) \to C^\infty(\overline{\mathbf{R}^+});\ f \to f|\overline{\mathbf{R}^+};\ \overline{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}.$$

We define $H^+ = F(\Phi(\overline{\mathbf{R}^+}));\ H_0^- = F(\Phi(\overline{\mathbf{R}^-}))$ which are orthogonal to each other. We have the following property: $h \in H^+$ (H_0^-) iff $h \in C^\infty(\mathbf{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{\mathrm{Im}\xi < 0\}$ ($\{\mathrm{Im}\xi > 0\}$) such that for all nonnegative integer l,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l}(\frac{c_k}{\xi^k})$$

as $|\xi| \to +\infty, \mathrm{Im}\xi \leq 0$ ($\mathrm{Im}\xi \geq 0$).

Let H' be the space of all polynomials and $H^- = H_0^- \bigoplus H';\ H = H^+ \bigoplus H^-$. Denote by π^+ (π^-) respectively the projection on H^+ (H^-). For calculations, we take $H = \tilde{H} = \{$rational functions having no poles on the real axis$\}$ ($\tilde{H}$ is a dense set in the topology of H). Then on $\tilde{H}$,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \to 0^-} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \tag{2.1}$$

where Γ^+ is a Jordan close curve included $\mathrm{Im}\xi > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. Similarly, define π' on $\tilde{H}$,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) d\xi. \tag{2.2}$$

So, $\pi'(H^-) = 0$. For $h \in H \bigcap L^1(R)$, $\pi' h = \frac{1}{2\pi} \int_R h(v)dv$ and for $h \in H^+ \bigcap L^1(R)$, $\pi' h = 0$.

An operator of order $m \in \mathbf{Z}$ and type d is a matrix

$$A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} : \begin{array}{c} C^\infty(X, E_1) \\ \bigoplus \\ C^\infty(\partial X, F_1) \end{array} \longrightarrow \begin{array}{c} C^\infty(X, E_2) \\ \bigoplus \\ C^\infty(\partial X, F_2) \end{array}.$$

where X is a manifold with boundary ∂X and E_1, E_2 (F_1, F_2) are vector bundles over X (∂X). Here, $P : C_0^\infty(\Omega, \overline{E_1}) \to C^\infty(\Omega, \overline{E_2})$ is a classical pseudodifferential operator of order m on Ω, where Ω is an open neighborhood of X and $\overline{E_i}|X = E_i$ ($i = 1, 2$). P has an extension: $\mathcal{E}'(\Omega, \overline{E_1}) \to \mathcal{D}'(\Omega, \overline{E_2})$, where $\mathcal{E}'(\Omega, \overline{E_1})$ ($\mathcal{D}'(\Omega, \overline{E_2})$) is the dual space of $C^\infty(\Omega, \overline{E_1})$ ($C_0^\infty(\Omega, \overline{E_2})$). Let $e^+ : C^\infty(X, E_1) \to \mathcal{E}'(\Omega, \overline{E_1})$ denote extension by zero from X to Ω and $r^+ : \mathcal{D}'(\Omega, \overline{E_2}) \to \mathcal{D}'(\Omega, E_2)$ denote the restriction from Ω to X, then define

$$\pi^+ P = r^+ P e^+ : C^\infty(X, E_1) \to \mathcal{D}'(\Omega, E_2). \tag{2.3}$$

In addition, P is supposed to have the transmission property; this means that, for all j,k,α, the homogeneous component p_j of order j in the asymptotic expansion of the symbol p of P in local coordinates near the boundary satisfies:

$$\partial^k_{x_n}\partial^\alpha_{\xi'}p_j(x',0,0,+1)=(-1)^{j-|\alpha|}\partial^k_{x_n}\partial^\alpha_{\xi'}p_j(x',0,0,-1),$$

then $\pi^+P:C^\infty(X,E_1)\to C^\infty(X,E_2)$ by [12]. Let G,T be respectively the singular Green operator and the trace operator of order m and type d. K is a potential operator and S is a classical pseudodifferential operator of order m along the boundary (For detailed definition, see [11]). Denote by $B^{m,d}$ the collection of all operators of order m and type d, and $\mathcal{B}$ is the union over all m and d.

Recall $B^{m,d}$ is a Fréchet space. The composition of the above operator matrices yields a continuous map: $B^{m,d}\times B^{m',d'}\to B^{m+m',\max\{m'+d,d'\}}$. Write

$$A=\left(\begin{array}{lr}\pi^+P+G & K\\ T & S\end{array}\right)\in B^{m,d},\ A'=\left(\begin{array}{lr}\pi^+P'+G' & K'\\ T' & S'\end{array}\right)\in B^{m',d'}.$$

The composition AA' is obtained by multiplication of the matrices(For more details see [12]). For example $\pi^+P\circ G'$ and $G\circ G'$ are singular Green operators of type d' and

$$\pi^+P\circ\pi^+P'=\pi^+(PP')+L(P,P'). \tag{2.4}$$

Here PP' is the usual composition of pseudodifferential operators and $L(P,P')$ called leftover term is a singular Green operator of type $m'+d$. The composition formulas of the above operator symbols will be given in the following.

2.2 Noncommutative Residue for Manifolds with Boundary

We assume that $E_1=E_2=E$; $F_1=F_2=F$ and $b(x',\xi',\xi_n,\eta_n)$ is the symbol of a singular Green operator G (about the definitions of symbols, see [11, p.11]), then

$$\mathrm{tr}(b)=\frac{1}{2\pi}\int_{\Gamma^+}b(x',\xi',\xi_n,\xi_n)d\xi_n=\bar{b}(x',\xi') \tag{2.5}$$

is a symbol on Y and $\bar{b}_{1-n}$ is obtained from b_{-n} (see [5]). Let $\mathbf{S}$ ($\mathbf{S}'$) be the unit sphere about ξ (ξ') and $\sigma(\xi)$ ($\sigma(\xi')$) be the corresponding canonical $n-1$ ($n-2$) volume form. Now we recall the main theorem in [5],

Theorem (Fedosov-Golse-Leichtnam-Schrohe) *Let X and ∂X be connected, $\dim X=n\geq 3$, $A=\left(\begin{array}{lr}\pi^+P+G & K\\ T & S\end{array}\right)\in B$, and denote by p, b and s the local symbols of P,G and S respectively. Define:*

$$\widetilde{\mathrm{Wres}}(A)=\int_X\int_{\mathbf{S}}\mathrm{tr}_E\left[p_{-n}(x,\xi)\right]\sigma(\xi)dx$$
$$+2\pi\int_{\partial X}\int_{\mathbf{S}'}\left\{\mathrm{tr}_E\left[(\mathrm{tr}b_{-n})(x',\xi')\right]+\mathrm{tr}_F\left[s_{1-n}(x',\xi')\right]\right\}\sigma(\xi')dx', \tag{2.6}$$

Then a) $\widetilde{\mathrm{Wres}}([A,B])=0$, for any $A,B\in\mathcal{B}$; b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

3 Properties of $(\Omega_{n,\pi^+S,X}(f_1,f_2),\Omega_{n-1,\pi^+S,Y}(f_1,f_2))$

Let X be a compact n-dimensional manifold with boundary Y and $\widetilde{X}=X\bigcup_Y X$. For a pseudodifferential operator S of order 0 with the transmission property acting on the sections of a vector bundle E over $\tilde{X}$, we consider the composition:

$$\widetilde{P}=\begin{pmatrix}\pi^+f_0 & 0\\ 0 & 0\end{pmatrix}\left[\begin{pmatrix}\pi^+S & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}\pi^+f_1 & 0\\ 0 & 0\end{pmatrix}\right]\left[\begin{pmatrix}\pi^+S & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}\pi^+f_2 & 0\\ 0 & 0\end{pmatrix}\right]$$
$$:=\pi^+f_0[\pi^+S,\pi^+f_1][\pi^+S,\pi^+f_2].$$

with $f_0,f_1,f_2\in C^\infty(X)$ which is the set $\{f|_X|f\in C^\infty(\tilde{X})\}$. By Section 2, $\pi^+S:C^\infty(X,E|_X)\to C^\infty(X,E|_X)$ is well defined and $\pi^+f_i:C^\infty(X,E|_X)\to C^\infty(X,E|_X)$ is just the multiplication by f_i for $i=0,1,2$ and $\widetilde{P}=\pi^+(f_0[S,f_1][S,f_2])+G$ where G is some singular Green operator. By (2.6),

$$\widetilde{\mathrm{Wres}}(\widetilde{P})=\int_X f_0\mathrm{wres}[S,\bar{f}_1][S,\bar{f}_2]|_X+2\pi\int_Y\mathrm{wres}_{x'}\mathrm{tr}(b).\tag{3.1}$$

Here $\bar{f}_1,\bar{f}_2$ are the extensions on $\widetilde{X}$ of f_1,f_2 and

$$\mathrm{wres}[S,\bar{f}_1][S,\bar{f}_2]=\int_{\mathbf{S}}\mathrm{tr}_E p_{-n}(x,\xi)\sigma(\xi)dx;\ \mathrm{wres}_{x'}\mathrm{tr}(b)=\int_{\mathbf{S'}}\mathrm{tr}_E(\mathrm{tr}b_{-n})(x',\xi')\sigma(\xi')dx',\tag{3.2}$$

where p_{-n},b_{-n} are respectively the order $-n$ symbols of $[S,\bar{f}_1][S,\bar{f}_2]$ and G. Write:

$$\Omega_n(f_1,f_2)=\mathrm{wres}[S,\bar{f}_1][S,\bar{f}_2]|_X=\Omega_n(\bar{f}_1,\bar{f}_2)|_X;\tag{3.3}$$

$$f_0|_Y\Omega_{n-1}(f_1,f_2)=2\pi\mathrm{wres}_{x'}\mathrm{tr}(b)\qquad(3.4),$$

then we have

$$\widetilde{\mathrm{Wres}}(\pi^+f_0[\pi^+S,\pi^+f_1][\pi^+S,\pi^+f_2])=\int_X f_0\Omega_n(f_1,f_2)+\int_Y f_0|_Y\Omega_{n-1}(f_1,f_2).\tag{3.5}$$

By [14], we have: $\Omega_n(f_1,f_2)=$

$$\int_{|\xi|=1}\mathrm{tr}\left[\sum\frac{1}{\alpha'!\alpha''!\beta!\delta!}D_x^\beta(\bar{f}_1)D_x^{\alpha''+\delta}(\bar{f}_2)\times\partial_\xi^{\alpha'+\alpha''+\beta}(\sigma_{-j}^S)\partial_\xi^\delta D_x^{\alpha'}(\sigma_{-k}^S)\right]\sigma(\xi)d^nx\,|_X\ ,\tag{3.6}$$

where σ_{-j}^S denotes the order $-j$ symbol of S; $D_x^\beta=(-i)^{|\beta|}\partial_x^\beta$ and the sum is taken over $|\alpha'|+|\alpha''|+|\beta|+|\delta|+j+k=n;|\beta|\geq1,|\delta|\geq1;\alpha',\alpha'',\beta,\delta\in\mathbf{Z}_+^n;j,k\in\mathbf{Z}_+$. By (3.6), this is a global n-form which is independent of the extensions of f_1,f_2.

Subsequently, we discuss the existence and uniqueness of $\Omega_{n-1}(f_1,f_2)$.

Recall, for example see [5, p.26], if A_1 and A_2 are pseudodifferential operators with the transmission property, then the Green operator

$$G=L(A_2,A_1)=\pi^+A_2\circ\pi^+A_1-\pi^+(A_2\circ A_1)$$

has a symbol b_{a_2,a_1}. If A_1 and A_2 have symbols $a_1(\eta_n) = a_1(x', x_n, \xi', \eta_n)$ and $a_2(\xi_n) = a_2(x', x_n, \xi', \xi_n)$ respectively, then b_{a_2,a_1} has an asymptotic expansion formula:

$$b_{a_2,a_1}(x', \xi', \xi_n, \eta_n) \sim \sum_{j,l,m=0}^{\infty} \frac{(-1)^m i^{j+l+m}}{j!l!m!} \partial_{\xi_n}^j \partial_{\eta_n}^m b_{\partial_{x_n}^j a_2|x_n=0, \partial_{\eta_n}^l \partial_{x_n}^{l+m} a_1|x_n=0}(x', \xi', \xi_n, \eta_n). \tag{3.7}$$

When a_1, a_2 are independent of x_n near the boundary , then we have:

$$b_{a_2,a_1}(x', \xi', \xi_n, \eta_n) = \frac{1}{2\pi} \int_{\Gamma^+} \frac{a_2^+(v) - a_2^+(\xi_n)}{v - \xi_n} \circ' \frac{a_1^-(\eta_n) - a_1^-(v)}{\eta_n - v} dv, \tag{3.8}$$

where $a_i^+(v) = \pi_v^+ a_i(x', 0, \xi', v), a_i^-(v) = \pi_v^- a_i(x', 0, \xi', v), i = 1, 2$ and

$$f(x', \xi', \xi_n) \circ' g(x', \xi', \eta_n) = \sum_{|\alpha| \geq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_{\xi'}^\alpha f \partial_{x'}^\alpha g. \tag{3.9}$$

Since $\pi_v^+ f(x) = 0$ and $\pi_v^- f(x) = f(x)$, we get $b_{a_2,a_1} = 0$ if a_1 or $a_2 = f(x)$ by (3.7) and (3.8). So $b_{\sigma(S),f_1} = 0$ and $[\pi^+ S, \pi^+ f] = \pi^+[S, f]$, then

$$\begin{aligned}
\pi^+ f_0[\pi^+ S, \pi^+ f_1][\pi^+ S, \pi^+ f_2] &= \pi^+ f_0 \pi^+[S, f_1] \pi^+[S, f_2] \\
&= \pi^+ f_0[\pi^+([S, f_1][S, f_2]) + \pi' B] \\
&= \pi^+(f_0[S, f_1][S, f_2]) + \pi^+ f_0 \circ \pi' B,
\end{aligned}$$

where $\pi' B = L([S, f_1], [S, f_2])$ (here we use f_i instead of $\bar{f}_i$) whose symbol is b.

In the following we assume that f_0 is independent of x_n near the boundary, then we have

$$\sigma_{-n}(\pi^+ f_0 \circ \pi' B) = f_0(x', 0) b_{-n}(x', \xi', \xi_n, \eta_n). \tag{3.10}$$

We can see it in the boundary chart by the equality (see [11, p.11])

$$(\pi^+ f_0 \circ \pi' B) u(x', x_n) = (2\pi)^{-n} \int e^{ix\xi} \Pi'_{\eta_n}[f_0(x') b(x', \xi', \xi_n, \eta_n)(e^+ u)^\wedge(\xi', \eta_n)] d\xi. \tag{3.11}$$

By definition:

$$\begin{aligned}
2\pi \mathrm{wres}_{x'} \mathrm{tr}(b) &= \int_{|\xi'|=1} 2\pi \mathrm{tr}\left[\mathrm{tr}\sigma_{-n}(\pi^+ f_0 \circ \pi' B)(x', \xi')\right] \sigma(\xi') d^{n-1} x' \\
&= f_0(x', 0) \int_{|\xi'|=1} \int_{\Gamma^+} \mathrm{tr} b_{-n}(x', \xi', \xi_n, \xi_n) d\xi_n \sigma(\xi') d^{n-1} x' \\
&= f_0|_Y \Omega_{n-1}(f_1, f_2), \qquad (3.12)
\end{aligned}$$

then

$$\Omega_{n-1}(f_1, f_2) = \int_{|\xi'|=1} \int_{\Gamma^+} \mathrm{tr} b_{-n}(x', \xi', \xi_n, \xi_n) d\xi_n \sigma(\xi') d^{n-1} x' \tag{3.13}$$

is an $(n-1)$-form over Y.

Theorem 3.1 *For the fixed S, the form pair $(\Omega_n(f_1, f_2), \Omega_{n-1}(f_1, f_2))$ is uniquely*

determined by (3.5) and (3.6).

Proof. $\Omega_n(f_1, f_2)$ is uniquely determined by (3.6). We assume that $\Omega'_{n-1}(f_1, f_2)$ also satisfies (3.5), then

$$\int_Y f_0|_Y \Omega_{n-1}(f_1, f_2) = \int_Y f_0|_Y \Omega'_{n-1}(f_1, f_2)$$

for any $f_0|_Y \in C^\infty(Y)$. (In fact, using a cut function, for any $g \in C^\infty(Y)$, there exists a function $f \in C^\infty(X)$ such that $f|_Y = g$ and f is independent of x_n near the boundary.) So $\Omega_{n-1}(f_1, f_2) = \Omega'_{n-1}(f_1, f_2)$. $\square$

Proposition 3.2 $\widetilde{\mathrm{Wres}}(\pi^+ f_0 \pi^+[S, f_1]\pi^+[S, f_2])$ *is a Hochschild 2-cocycle (for definition, see* [6]*) over* $C^\infty(X)$.

Proof. This proposition comes from the relations:
$[S, fh] = [S, f]h + f[S, h]$; $\pi^+(f_1[S, f_2]) = \pi^+ f_1 \pi^+[S, f_2]$; $\pi^+ f_0 \pi^+ f_1 = \pi^+(f_1 f_0)$ and the trace property of $\widetilde{\mathrm{Wres}}$. $\square$

Remark: $\int_X f_0 \Omega_n(f_1, f_2)$ and $\int_Y f_0|_Y \Omega_{n-1}(f_1, f_2)$ are not Hochschild 2-cocycle over $C^\infty(X)$.

Proposition 3.3 $\Omega_n(f_1, f_2)$ *and* $\Omega_{n-1}(f_1, f_2)$ *are symmetric in* f_1 *and* f_2.

Proof. By [14], $\Omega_n(\overline{f_1}, \overline{f_2})$ is symmetric in $\overline{f_1}$ and $\overline{f_2}$, so $\Omega_n(f_1, f_2)$ is symmetric in f_1 and f_2. By the trace property of $\widetilde{\mathrm{Wres}}$ and the commutativity of $C^\infty(X)$, we note that:

$$\widetilde{\mathrm{Wres}}(\pi^+ f_0 \pi^+[S, f_1]\pi^+[S, f_2] = \widetilde{\mathrm{Wres}}(\pi^+ f_0 \pi^+[S, f_2]\pi^+[S, f_1])$$

So $\Omega_{n-1}(f_1, f_2)$ is also symmetric in f_1, f_2 by (3.5). $\square$

Remark: The condition "$fS^2 = S^2 f$" in the theorem 2.7 of [14] is not used here.

In the following, we write the expression of $\Omega_{n-1}(f_1, f_2)$ in detail. Let:

$$\overline{b_{a_1,a_2}} := \mathrm{tr}(b_{a_1,a_2}) = \frac{1}{2\pi} \int_{\Gamma^+} b_{a_1,a_2}(x', \xi', \xi_n, \xi_n) d\xi_n. \tag{3.14}$$

By [5, p.27], we have the formula:

$$\overline{b_{a_1,a_2}} = \sum_{j,k=0}^{\infty} \frac{(-i)^{j+k+1}}{(j+k+1)!} \pi'_{\xi_n} \left[\partial^j_{x_n} \partial^k_{\xi_n} a_1^+(x', 0, \xi', \xi_n) \circ' \partial^{j+1}_{\xi_n} \partial^k_{x_n} a_2^-(x', 0, \xi', \xi_n) \right]. \tag{3.15}$$

Using (2.2),(3.13),(3.14) and (3.15), one obtains:

$$\Omega_{n-1}(f_1, f_2) = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left\{ \mathrm{trace} \sum_{j,k=0}^{\infty} \frac{(-i)^{j+k+1}}{(j+k+1)!} \right.$$
$$\left. \times \left[\partial^j_{x_n} \partial^k_{\xi_n} a_1^+(x', 0, \xi', \xi_n) \circ' \partial^{j+1}_{\xi_n} \partial^k_{x_n} a_2(x', 0, \xi', \xi_n) \right]_{-n} \right\} d\xi_n \sigma(\xi') d^{n-1}x' \tag{3.16}$$

since the $+ +$ parts vanish after integration with respect to ξ_n (see [5, p.23]).

For $\pi' B = L([S, f_1], [S, f_2])$, by [14] lemma 2.2, then for $i = 1, 2$, we have:

$$a_i = \sigma[S, f_i] = \sum_{k \geq 1} \sigma_{-k}[S, f_i] = \sum_{k \geq 1} \left[\sum_{|\beta|=1}^{k} \frac{1}{\beta!} D_x^\beta(f_i) \partial_\xi^\beta(\sigma^S_{-(k-|\beta|)}) \right]. \tag{3.17}$$

By (3.9), then:

$$\left[\partial_{x_n}^j \partial_{\xi_n}^k a_1^+(x',0,\xi',\xi_n) \circ' \partial_{\xi_n}^{j+1}\partial_{x_n}^k a_2(x',0,\xi',\xi_n)\right]_{-n}$$

$$\begin{aligned}
&= \left[\sum_{r,l} \partial_{x_n}^j \partial_{\xi_n}^k a_{1(r)}^+(x',0,\xi',\xi_n) \circ' \partial_{\xi_n}^{j+1}\partial_{x_n}^k a_{2(l)}(x',0,\xi',\xi_n)\right]_{-n} \\
&= \left[\sum_{r,l}\sum_{|\alpha|\geq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k a_{1(r)}^+(x',0,\xi',\xi_n) \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1}\partial_{x_n}^k a_{2(l)}(x',0,\xi',\xi_n)\right]_{-n} \\
&= \sum \frac{(-i)^{|\alpha|}}{\alpha!} \partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k a_{1(r)}^+(x',0,\xi',\xi_n) \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1}\partial_{x_n}^k a_{2(l)}(x',0,\xi',\xi_n)
\end{aligned}$$

where the sum is taken over $r-k-|\alpha|+l-j-1=-n, \;\; r,l\leq -1, \;\; |\alpha|\geq 0$ for the fixed j,k and $a_{1(r)}^+$ ($a_{2(l)}$) denotes the order r (l) symbol of a_1^+ (a_2). Using:

$$a_{1(r)}^+ = \pi_{\xi_n}^+ a_{1(r)} = \pi_{\xi_n}^+ \left[\sum_{|\beta|=1}^{-r} \frac{(-i)^{|\beta|}}{\beta!} \partial_x^\beta (f_1) \partial_\xi^\beta (\sigma_{r+|\beta|}^S)\right] = \sum_{|\beta|=1}^{-r} \frac{(-i)^{|\beta|}}{\beta!} \partial_x^\beta (f_1) \pi_{\xi_n}^+ \partial_\xi^\beta (\sigma_{r+|\beta|}^S);$$

$$a_{2(l)} = \sum_{|\delta|=1}^{-l} \frac{(-i)^{|\delta|}}{\delta!} \partial_x^\delta (f_2) \partial_\xi^\delta (\sigma_{l+|\delta|}^S),$$

we have:

$$\left[\partial_{x_n}^j \partial_{\xi_n}^k a_1^+(x',0,\xi',\xi_n) \circ' \partial_{\xi_n}^{j+1}\partial_{x_n}^k a_2(x',0,\xi',\xi_n)\right]_{-n}$$

$$= \sum \sum_{|\beta|=1}^{-r} \sum_{|\delta|=1}^{-s} \frac{(-i)^{|\alpha|+|\beta|+|\delta|}}{\alpha!\beta!\delta!} \partial_{x_n}^j \left[\partial_x^\beta (f_1) \partial_{\xi'}^\alpha \partial_{\xi_n}^k \pi_{\xi_n}^+ \partial_\xi^\beta (\sigma_{r+|\beta|}^S)\right]|_{x_n=0} \times$$

$$\partial_{x'}^\alpha \partial_{x_n}^k \left[\partial_x^\delta (f_2) \partial_{\xi_n}^{j+1} \partial_\xi^\delta \sigma_{(l+|\delta|)}^S\right]|_{x_n=0} \tag{3.18}$$

with the sum $\sum$ as before. By (3.16) and (3.18), we get:

$$\Omega_{n-1}(f_1,f_2) = \sum_{j,k=0}^{\infty} \sum \sum_{|\beta|=1}^{-r} \sum_{|\delta|=1}^{-l} \frac{(-i)^{j+k+1+|\alpha|+|\beta|+|\delta|}}{\alpha!\beta!\delta!(j+k+1)!}$$

$$\times \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \mathrm{trace}\left\{\partial_{x_n}^j \left[\partial_x^\beta (f_1) \partial_{\xi'}^\alpha \partial_{\xi_n}^k \pi_{\xi_n}^+ \partial_\xi^\beta (\sigma_{r+|\beta|}^S)\right]|_{x_n=0}\right.$$

$$\left. \times \partial_{x'}^\alpha \partial_{x_n}^k \left[\partial_x^\delta (f_2) \partial_{\xi_n}^{j+1} \partial_\xi^\delta \sigma_{(l+|\delta|)}^S\right]|_{x_n=0}\right\} d\xi_n \sigma(\xi') d^{n-1}x' \tag{3.19}$$

with the sum $\sum$ as (3.18).

4 The Even Dimensional Case

Let (X, g) be an even dimensional, compact, oriented, Riemannian manifold with boundary Y and product metric near the boundary. $(\widetilde{X}, \widetilde{g})$ is the associated double manifold. Let $(E, S) = (H, F)$ associated to $(\widetilde{X}, \widetilde{g})$ introduced by Section 1. Let the dimension of X be n. Since $\int_{|\xi|=1}\{$the product of odd number of ξ_i $\}\sigma(\xi) = 0$, then we have

Lemma 4.1 *$\Omega_n(f_1, f_2) = 0$ when n is odd and $\Omega_{n-1}(f_1, f_2) = 0$ when n is even.*

Since n is even, by (3.5) and Lemma 4.1,we get:

$$\widetilde{\mathrm{Wres}}(\pi^+ f_0[\pi^+ F, \pi^+ f_1][\pi^+ F, \pi^+ f_2]) = \int_X f_0 \Omega_n(f_1, f_2). \tag{4.1}$$

Definition 4.2 A subconformal manifold is an equivalence of Riemannian manifolds. Two metrics g and $\widetilde{g}$ are said to be equivalent if $\widetilde{g} = e^\eta g$, where η satisfies $\star)$ condition i.e. $\eta \in C^\infty(X)$; $\eta \bigcup \eta \in C^\infty(\widetilde{X})$ where $\eta \bigcup \eta = \eta$ on both copies of X.

Example: 1) $X = \mathbf{R}^n_+$ and $f(x)$ is an even function about x_n, take $f|\mathbf{R}^n_+ = \eta$, then $\eta \bigcup \eta$ satisfies $\star)$ condition.

2) $f(x)$ is independent of x_n near the boundary.

3) $f(x) \in C^\infty(X), f(x) = e^{\frac{1}{x_n^2 - 1}} f(x')$ near the boundary and if not, $f(x) = 0$.

Since the smoothness of $\eta \bigcup \eta$ just depends on a neighborhood of the boundary, so we get:

Proposition 4.3 *$\eta \in C^\infty(X)$ satisfies $\star)$ condition iff $\exists f \in C^\infty(\widetilde{X})$ such that $f|_{Y\times(-1,1)} = \eta \bigcup \eta|_{Y\times(-1,1)}$.*

Proposition 4.4 *$\Omega_n(f_1, f_2)$ is subconformally invariant for the above subconformal manifold.*

Proof: Let $\widetilde{g} = e^\eta g$, where η satisfies $\star)$ condition, so $\widetilde{g} \bigcup \widetilde{g} = e^{\eta \bigcup \eta} g \bigcup g$ and $\eta \bigcup \eta \in C^\infty(\widetilde{X})$. By [3] or [14] $\Omega_n(\bar{f}_1, \bar{f}_2)$ is conformal invariant, then $\Omega_{n, g\bigcup g}(\bar{f}_1, \bar{f}_2) = \Omega_{n, \widetilde{g}\bigcup\widetilde{g}}(\bar{f}_1, \bar{f}_2)$ and $\Omega_{n,g}(f_1, f_2) = \Omega_{n,\widetilde{g}}(f_1, f_2)$, where $\Omega_{n,g}(f_1, f_2)$ denotes $\Omega_n(f_1, f_2)$ associated to g. □

By [2, p.339], we have

Theorem 4.5 *Let $[(X, g)]$ be a 4-dimensional subconformal manifold with boundary as in the definition 4.2 and $[(\widetilde{X}, \widetilde{g})]$ be the associated subconformal manifold without boundary, then*

$$\Omega_4(f_1, f_2) = \frac{1}{16\pi^2}\left[\frac{1}{3} r \langle d\widetilde{f_1}, d\widetilde{f_2}\rangle - \triangle\langle d\widetilde{f_1}, d\widetilde{f_2}\rangle + \langle \nabla d\widetilde{f_1}, \nabla d\widetilde{f_2}\rangle - \frac{1}{2}(\triangle\widetilde{f_1})(\triangle\widetilde{f_2})\right] \mathrm{Vol}|_X, \tag{4.2}$$

where $\widetilde{f_1}, \widetilde{f_2} \in C^\infty(\widetilde{X})$ are the extensions of f_1, f_2, r the scalar curvature, Vol the volume form on $\widetilde{X}$, $\triangle$ the Laplacian and ∇ the Levi-civita connection associated to any metric of $[(\widetilde{X}, \widetilde{g})]$.

5 $\Omega_{n-1}(f_1, f_2)$ for Flat Manifolds in the x_n-Independent Case

In the rest of this paper, (X, g) always denotes an odd dimensional, compact, oriented Riemannian manifold with boundary Y and product metric near the boundary. Similar to Section 4, we let $(E, S) = (L^2(\wedge_c^{\frac{n+1}{2}} T^\star \widetilde{X}), F)$, then $\Omega_n(f_1, f_2) = 0$ by Lemma 4.1. So for f_0 independent of x_n near the boundary, we have

$$\widetilde{\mathrm{Wres}}(\pi^+ f_0[\pi^+ F, \pi^+ f_1][\pi^+ F, \pi^+ f_2]) = \int_Y (f_0|_Y)\Omega_{n-1}(f_1, f_2). \tag{5.1}$$

In this section, we assume that X is flat and f_1, f_2 are independent of x_n near the boundary and write $\Omega_{n-1,\mathrm{flat}}(f_1, f_2)$ instead of $\Omega_{n-1}(f_1, f_2)$.

We follow the method in Section 4 in [14]. Since X is flat, so is $(\widetilde{X}, \widetilde{g})$. Then by Proposition 3.1 in [14], we have $\sigma(F) = \sigma_L(F)$ is independent of x where $\sigma(F)$ $(\sigma_L(F))$ is the symbol (leading symbol) of F. Using this information we deduce from (3.19) $j = k = 0$ and $|\beta| = -r, |\delta| = -l$. Let $\beta = (\beta', \beta''), \delta = (\delta', \delta'')$, where $\beta', \delta' \in Z_+^{n-1}$, $\beta'', \delta'' \in Z_+$, then by f_1, f_2 are independent of x_n near the boundary, we have $\beta'' = \delta'' = 0$ and

$$\Omega_{n-1,\mathrm{flat}}(f_1, f_2) = \sum \sum_{|\beta'|=-r} \sum_{|\delta|=-l} \frac{(-i)^{1+|\alpha|-r-s}}{\alpha!\beta'!\delta'!} \partial_{x'}^{\beta'} f_1(x', 0) \partial_{x'}^{\alpha+\delta'} f_2(x', 0) \times$$

$$\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \mathrm{trace}\left[\pi_{\xi_n}^+ \partial_{\xi'}^{\alpha+\beta'} \sigma_L(F) \times \partial_{\xi_n} \partial_{\xi'}^{\delta'} \sigma_L(F)\right] d\xi_n \sigma(\xi') d^{n-1}x',$$

where the sum is taken over $r + s - |\alpha| - 1 = -n, r \leq -1,\ s \leq -1,\ |\alpha| \geq 0$. We get

Lemma 5.1

$$\Omega_{n-1,\mathrm{flat}}(f_1, f_2) = \sum \frac{(-i)^n}{\alpha!\beta'!\delta'!} \partial_{x'}^{\beta'} f_1(x', 0) \partial_{x'}^{\alpha+\delta'} f_2(x', 0) \times$$

$$\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \mathrm{trace}\left[\pi_{\xi_n}^+ \partial_{\xi'}^{\alpha+\beta'} \sigma_L(F) \times \partial_{\xi_n} \partial_{\xi'}^{\delta'} \sigma_L(F)\right] d\xi_n \sigma(\xi') d^{n-1}x', \tag{5.2}$$

where the sum is taken over $|\beta'| + |\delta'| + |\alpha| = n - 1, |\beta'| \geq 1, |\delta'| \geq 1$.

To better handle the previous expression, we consider:

$$\phi(\xi', \xi_n, u, v) := \sum \frac{1}{\alpha!\beta'!\delta'!} u^{\beta'} v^{\alpha+\delta'} \mathrm{trace}\left[\pi_{\xi_n}^+ \partial_{\xi'}^{\alpha+\beta'} \sigma_L(F) \times \partial_{\xi'}^{\delta'} \partial_{\xi_n} \sigma_L(F)\right] \tag{5.3}$$

with the sum as before and $u, v \in \mathbf{R}^{n-1}$. Then by a recursive way we have:

$$\Omega_{n-1,\mathrm{flat}}(f_1, f_2) = (-i)^n \left[\sum A_{a,b} \partial_{x'}^a f_1(x', 0) \partial_{x'}^b f_2(x', 0)\right] d^{n-1}x'$$

where A_{ab} is a number satisfying $\sum A_{a,b} u^a v^b = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \phi(\xi', \xi_n, u, v) d\xi_n \sigma(\xi')$ and the sum is taken over $a + b = n - 1$ and $a \geq 1,\ b \geq 1,\ a, b \in Z_{n-1}^+$. Instead of a direct

approach to compute trace $\left[\pi^+_{\xi_n}\partial^{\alpha+\beta'}_{\xi'}\sigma_L(F)\times\partial^{\delta'}_{\xi'}\partial_{\xi_n}\sigma_L(F)\right]$, we shall use the Taylor expansion of function:

$$\psi(\xi',\eta',\xi_n) := \mathrm{trace}\left[\pi^+_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right].$$

Considering the Taylor expansion of $\psi(\xi'+u,\eta'+v,\xi_n)$ about u,v at $u=v=0$, then:

$$\psi(\xi'+u,\eta'+v,\xi_n) = \sum_{|\beta|\geq 0}\sum_{|\delta|\geq 0}\frac{u^\beta v^\delta}{\beta!\delta!}\mathrm{trace}\left[\partial^\beta_{\xi'}\pi^+_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial^\delta_{\eta'}\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right]$$

with $(\beta,\delta)=(\alpha_1,...,\alpha_{n-1},\alpha_n,...,\alpha_{2(n-1)})$.

Write $\psi(\xi',\eta',u,v,\xi_n) := \psi(\xi'+u,\eta'+v,\xi_n)$ and

$$T'_{n-1}\psi(\xi',\eta',u,v,\xi_n) := \sum\frac{u^\beta v^\delta}{\beta!\delta!}\mathrm{trace}\left[\partial^\beta_{\xi'}\pi^+_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial^\delta_{\eta'}\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right], \tag{5.4}$$

where the sum is taken over $|\beta|+|\delta|=n-1,\ |\beta|\geq 1,\ |\delta|\geq 1$ i.e. term of order $n-1$ in the Taylor expansion of $\psi(\xi'+u,\eta'+v,\xi_n)$ minus the terms with only powers of u or only powers of v. Now, write:

$$P = \mathrm{trace}\left[\partial^\beta_{\xi'}\pi^+_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial^\delta_{\eta'}\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right],$$

then:

$$T'_{n-1}\psi(\xi',\eta',u+v,v,\xi_n) = \sum\frac{(u+v)^\beta v^\delta}{\beta!\delta!} = \sum\sum_{\beta'+\beta''=\beta}\frac{u^{\beta'}v^{\beta''+\delta}}{\beta'!\beta''!\delta!}P$$

$$= \sum\sum_{\beta'+\beta''=\beta;\beta'\neq 0}\frac{u^{\beta'}v^{\beta''+\delta}}{\beta'!\beta''!\delta!}P + \sum\frac{v^{\beta+\delta}}{\beta!\delta!}P$$

where the sum $\sum$ is taken over $|\beta|+|\delta|=n-1;\ |\beta|\geq 1;\ |\delta|\geq 1$.

$$= \sum\frac{u^{\beta'}v^{\beta''+\delta}}{\beta'!\beta''!\delta!}P + T'_{n-1}\psi(\xi',\eta',v,v,\xi_n).$$

where the sum $\sum$ is taken over $|\beta'|+|\delta'|+|\delta|=n-1;\ |\beta'|\geq 1;\ |\delta|\geq 1$. Therefore, by taking $\eta=\xi$ we obtain:

$$T'_{n-1}\psi(\xi',\xi',u+v,v,\xi_n) - T'_{n-1}\psi(\xi',\xi',v,v,\xi_n) = \phi(\xi',\xi_n,u,v).$$

In summary, we have:

Theorem 5.2

$$\Omega_{n-1,\mathrm{flat}}(f_1,f_2) = (-i)^n\left[\sum A_{a,b}\partial^a_{x'}f_1(x',0)\partial^b_{x'}f_2(x',0)\right]d^{n-1}x', \tag{5.5}$$

where $\sum A_{a,b}u^a v^b = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\left[T'_{n-1}\psi(\xi',\xi',u+v,v,\xi_n) - T'_{n-1}\psi(\xi',\xi',v,v,\xi_n)\right]d\xi_n\sigma(\xi')$. *and* $T'_{n-1}\psi(\xi',\eta',u,v,\xi_n)$ *is defined by (5.4).*

By Theorem 5.2, to obtain an explicit expression of Ω_{n-1} in the flat case, it is necessary to study $\psi(\xi',\eta',\xi_n)$ for ξ' and η' not zero in $T_x^\star Y$. Recall the theorem 4.3 in [14] (we will find it is also correct when n is odd through the check.) says that: when $\sigma_L(F)(\xi)\sigma_L(F)(\eta)$ acts on m-forms on $\widetilde{X}$, then

$$\mathrm{trace}\left[\sigma_L(F)(\xi)\times\sigma_L(F)(\eta)\right]=a_{n,m}\frac{\langle\xi,\eta\rangle^2}{|\xi|^2|\eta|^2}+b_{n,m}\tag{5.6}$$

where $b_{n,m}=C_n^m-a_{n,m}=C_{n-2}^{m-2}+C_{n-2}^m-2C_{n-2}^{m-1}$ and C_n^m denotes a combinator number. Suppose that $g=g^Y+d^2x_n$ near the boundary, then

$$\langle\xi,\eta\rangle_g=\langle\xi',\eta'\rangle_{g^Y}+\xi_n\eta_n\tag{5.7}$$

where $\xi=\xi'+\xi_ndx_n;\ \eta=\eta'+\eta_ndx_n$. By (5.6) and (5.7), then

$$\begin{aligned}\mathrm{trace}[\pi^+_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)] &= \pi^+_{\xi_n}\partial_{\eta_n}\mathrm{trace}[\sigma_L(F)(\xi)\times\sigma_L(F)(\eta)]|_{\eta_n=\xi_n}\\ &= \pi^+_{\xi_n}\partial_{\eta_n}\left[a_{n,m}\frac{\langle\xi,\eta\rangle^2}{|\xi|^2|\eta|^2}+b_{n,m}\right]|_{\eta_n=\xi_n}\\ &= a_{n,m}\pi^+_{\xi_n}\left[\frac{2\langle\xi,\eta\rangle\xi_n|\eta|^2-2\eta_n\langle\xi,\eta\rangle^2}{|\xi|^2|\eta|^4}\right]|_{\eta_n=\xi_n},\end{aligned}$$

by (2.1), Cauchy integral formula and the choice of Γ^+:

$$\begin{aligned}&= \frac{a_{n,m}}{|\eta|^4}\frac{1}{2\pi i}\lim_{u\to0^-}\int_{\Gamma^+}\frac{2(\langle\xi',\eta'\rangle+z\eta_n)z|\eta|^2-2\eta_n(\langle\xi',\eta'\rangle+z\eta_n)^2}{(|\xi'|^2+z^2)(\xi_n+iu-z)}dz|_{\eta_n=\xi_n}\\ &= \frac{a_{n,m}}{|\eta|^4}\frac{2(\langle\xi',\eta'\rangle+i|\xi'|\eta_n)i|\xi'|\|\eta|^2-2\eta_n(\langle\xi',\eta'\rangle+i|\xi'|\eta_n)^2}{2i|\xi'|(\xi_n-i|\xi'|)}|_{\eta_n=\xi_n}\\ &= \frac{a_{n,m}(\langle\xi',\eta'\rangle+i|\xi'|\xi_n)}{i|\xi'|(\xi_n-i|\xi'|)(|\eta'|^2+\xi_n^2)^2}\left[|\eta'|^2i|\xi'|-\xi_n\langle\xi',\eta'\rangle\right].\end{aligned}$$

So we have:
Theorem 5.3 Suppose that (X,g) has a product metric near the boundary. When $\sigma_L(F)(\xi',\xi_n)\sigma_L(F)(\eta',\xi_n)$ acting on m-forms in the boundary chart, then

$$\begin{aligned}&\mathrm{trace}\left[\pi^+_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right]\\ &\qquad=\frac{a_{n,m}(\langle\xi',\eta'\rangle+i|\xi'|\xi_n)}{i|\xi'|(\xi_n-i|\xi'|)(|\eta'|^2+\xi_n^2)^2}\left[|\eta'|^2i|\xi'|-\xi_n\langle\xi',\eta'\rangle\right],\end{aligned}\tag{5.8}$$

where $a_{n,m}=C_n^m-C_{n-2}^{m-2}-C_{n-2}^m+2C_{n-2}^{m-1}$.

6 $\Omega_{n-1}(f_1,f_2)$ for Flat Manifolds in the x_n-Dependent Case

In this section, we assume that X is flat and f_1,f_2 are dependent of x_n near the boundary.

Since X is flat, so $\sigma(F)=\sigma_L(F)$ and $|\beta|=-r,|\delta|=-s$. By (3.19), we have:
Lemma 6.1

$$\Omega_{n-1,\mathrm{flat}}(f_1,f_2)=\sum_{j,k=0}^{\infty}\sum\frac{(-i)^n}{\alpha!\beta'!\beta''!\delta'!\delta''!(j+k+1)!}\partial_{x_n}^{j+\beta''}\partial_{x'}^{\beta'}f_1|_{x_n=0}\times\partial_{x'}^{\alpha+\delta'}\partial_{x_n}^{k+\delta''}f_2|_{x_n=0}\times$$

$$\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left\{\partial_{\xi'}^{\alpha+\beta'}\partial_{\eta'}^{\delta'}\left[\partial_{\xi_n}^{k}\pi_{\xi_n}^{+}\partial_{\xi_n}^{\beta''}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}^{j+1+\delta''}\sigma_L(F)(\eta',\xi_n)\right]|_{\xi'=\eta'}\right\}d\xi_n\sigma(\xi')d^{n-1}x', \tag{6.1}$$

where the sum is taken over $|\beta'|+\beta''+|\delta'|+\delta''+|\alpha|+j+k+1=n,\quad|\beta'|+\beta''\geq 1,\quad|\delta'|+\delta''\geq 1,\quad|\alpha|\geq 0.$

Similar to Section 5, we want to give its explicit expression. Let:

$$\widetilde{\phi}(\xi',\xi_n,u,v):=\sum\frac{1}{\alpha!\beta'!\beta''!\delta'!\delta''!(j+k+1)!}u^{(\beta',j+\beta'')}v^{(\alpha+\delta',k+\delta'')}$$

$$\times\mathrm{trace}\left[\partial_{\xi'}^{\alpha+\beta'}\partial_{\xi_n}^{k}\pi_{\xi_n}^{+}\partial_{\xi_n}^{\beta''}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi'}^{\delta'}\partial_{\xi_n}^{j+1+\delta''}\sigma_L(F)(\xi',\xi_n)\right]$$

with the sum as before and $u,v\in\mathbf{R}^n$. One obtains:

$$\Omega_{n-1,\mathrm{flat}}(f_1,f_2)=(-i)^n\left[\sum A_{a,b}\partial_x^a f_1(x',0)\partial_x^b f_2(x',0)\right]d^{n-1}x'$$

with $\sum A_{a,b}u^av^b=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\widetilde{\phi}(\xi',\xi_n,u,v)d\xi_n\sigma(\xi')$.
Now,

$$\widetilde{\phi}(\xi',\xi_n,u,v)=\sum_{j,k=0}^{\infty}\sum_{\beta'',\delta''}\frac{u_n^{j+\beta''}v_n^{k+\delta''}}{\beta''!\delta''!(j+k+1)!}\widetilde{\phi}_{j,k,\beta'',\delta''}(\xi',\eta',\xi_n,u',v')|_{\eta'=\xi'} \tag{6.2}$$

with

$$\widetilde{\phi}_{j,k,\beta'',\delta''}(\xi',\eta',\xi_n,u',v')=\sum\frac{1}{\alpha!\beta'!\delta'!}u'^{\beta'}v'^{\alpha+\delta'}$$

$$\times\partial_{\xi'}^{\alpha+\beta'}\partial_{\eta'}^{\delta'}\mathrm{trace}\left[\partial_{\xi_n}^{k}\pi_{\xi_n}^{+}\partial_{\xi_n}^{\beta''}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}^{j+1+\delta''}\sigma_L(F)(\eta',\xi_n)\right] \tag{6.3}$$

where the sum is taken over $|\beta'|+|\delta'|+|\alpha|=n-(j+k+1)-\beta''-\delta''=s,\ \ |\beta'|+\beta''\geq 1,\ \ |\delta'|+\delta''\geq 1$ for fixed j,k,β'',δ''.
Write

$$\psi_{j,k,\beta'',\delta''}(\xi',\eta',\xi_n):=\mathrm{trace}\left[\partial_{\xi_n}^{k}\pi_{\xi_n}^{+}\partial_{\xi_n}^{\beta''}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}^{j+1+\delta''}\sigma_L(F)(\eta',\xi_n)\right];$$

$$\psi_{j,k,\beta'',\delta''}(\xi',\eta',\xi_n,u',v'):=\psi_{j,k,\beta'',\delta''}(\xi'+u',\eta'+v',\xi_n);$$

$$T_s\psi_{j,k,\beta'',\delta''}(\xi',\eta',u',v',\xi_n):=\sum_{|\beta|+|\delta|=s}\frac{u'^{\beta}v'^{\delta}}{\beta!\delta!}$$

$$\times\mathrm{trace}\left[\partial_{\xi'}^{\beta}\partial_{\xi_n}^{k}\pi_{\xi_n}^{+}\partial_{\xi_n}^{\beta''}\sigma_L(F)(\xi',\xi_n)\times\partial_{\eta'}^{\delta}\partial_{\xi_n}^{j+1+\delta''}\sigma_L(F)(\eta',\xi_n)\right]$$

i.e. the term of order s in the Taylor expression of $\psi_{j,k,\beta'',\delta''}(\xi',\eta',u,v,\xi_n)$. By (6.3),

$$\widetilde{\phi}_{j,k,\beta''\neq 0,\delta''\neq 0}(\xi',\eta',\xi_n,u',v') = \sum_{|\beta'|+|\delta'|+|\alpha|=s} \frac{1}{\alpha!\beta'!\delta'!} u'^{\beta'} v'^{\alpha+\delta'} \times \partial_{\xi'}^{\alpha+\beta'} \partial_{\eta'}^{\delta'} \psi_{\beta''\neq 0,\delta''\neq 0}(\xi',\eta',\xi_n)$$

$$\begin{aligned} T_s\psi_{\beta''\neq 0,\delta''\neq 0}(\xi',\eta',u'+v',v') &= \sum_{|\beta|+|\delta|=s} \sum_{\beta'+\beta''=\beta} \frac{u'^{\beta'} v'^{\beta''+\delta}}{\beta'!\beta''!\delta!} \partial_{\xi'}^{\beta'+\beta"} \partial_{\eta'}^{\delta} \psi_{j,k,\beta''\neq 0,\delta''\neq 0} \\ &= \widetilde{\phi}_{\beta''\neq 0,\delta''\neq 0}(\xi',\eta',u',v',\xi_n). \qquad (6.4) \end{aligned}$$

Let $T_{s,u}(T_{s,v})$ denote the term of order s in the Tayler expansion $\psi_{j,k,\beta'',\delta''}$ minus the terms with only powers of v (u) and T'_s denote the term of order s in the Tayler expansion $\psi_{j,k,\beta'',\delta''}$ minus the terms with only powers of u or v. In a similar way, we get:

$$\begin{aligned} \widetilde{\phi}_{\beta''=0,\delta''=0}(\xi',\eta',u',v',\xi_n) &= T'_s\psi_{\beta''=0,\delta''=0}(\xi',\eta',u'+v',v') - T'_s\psi_{\beta''=0,\delta''=0}(\xi',\eta',v',v'); \\ \widetilde{\phi}_{\beta''=0,\delta''\neq 0}(\xi',\eta',u',v',\xi_n) &= T_{s,u}\psi_{\beta''=0,\delta''\neq 0}(\xi',\eta',u'+v',v') - T_{s,u}\psi_{\beta''=0,\delta''\neq 0}(\xi',\eta',v',v'); \\ \widetilde{\phi}_{\beta''\neq 0,\delta''=0}(\xi',\eta',u',v',\xi_n) &= T_{s,v}\psi_{\beta''\neq 0,\delta''=0}(\xi',\eta',u'+v',v') - T_{s,v}\psi_{\beta''\neq 0,\delta''=0}(\xi',\eta',v',v'). \end{aligned} \qquad (6.5)$$

Summarizing, we have a similar result for manifolds with boundary to the theorem 4.2 in [14]:

Theorem 6.2

$$\Omega_{n-1,\mathrm{flat}}(f_1,f_2) = (-i)^n \left[\sum A_{a,b}\partial_x^a f_1(x',0)\partial_x^b f_2(x',0)\right] d^{n-1}x'$$

with $\sum A_{a,b}u^a v^b = \int_{|\xi'|=1}\int_{-\infty}^{+\infty} \widetilde{\phi}(\xi',\xi_n,u,v)d\xi_n\sigma(\xi')$ *and* $\widetilde{\phi}(\xi',\xi_n,u,v)$ *is determined by (6.2) (6.4) and (6.5).*

The computation of $\psi_{j,k,\beta'',\delta''}(\xi',\eta',\xi_n)$ is similar to the theorem 5.3.

7 The Computation of $\Omega_2(f_1,f_2)$ when $n=3$

In this section, we will give the global expression of $\Omega_2(f_1,f_2)$ in three cases.

a) Flat and f_1,f_2 Are Independent of x_n Near the Boundary.

By lemma 5.1 and $n=3$, we have $|\delta'|=|\beta'|=1$, $|\alpha|=0$ and

$$\Omega_{2,\mathrm{flat}}(f_1,f_2) = \sum_{i,j=1}^{2}(-i)^3\partial_{x_i}f_1(x',0)\partial_{x_j}f_2(x',0)$$

$$\times \int_{|\xi'|=1}\int_{-\infty}^{+\infty} \partial_{\xi_i}\partial_{\eta_j}\left\{\mathrm{trace}\left[\pi^+_{\xi_3}\sigma_L(F)\times\partial_{\xi_3}\sigma_L(F)\right]\right\}|_{\xi'=\eta'}d\xi_3\sigma(\xi')dx_1\wedge dx_2.$$

In this subsection we denote $\Big|_{|\xi'|=1}^{\xi'=\eta'}$ by $|_\star$. Using the theorem 5.3, then for $n=3$, $m=2$ we have:

$$D_{ij} := \partial_{\xi_i}\partial_{\eta_j}\left\{\mathrm{trace}\left[\pi^+_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right]\right\}|_\star$$

$$= \quad \partial_{\xi_i}\left(\frac{a_{n,m}}{\xi_n - i|\xi'|}A\right)|_\star = \left[\frac{i a_{n,m}\xi_i}{(\xi_n - i|\xi'|)^2|\xi'|}A + \frac{a_{n,m}}{\xi_n - i|\xi'|}\partial_{\xi_i}A\right]|_\star,$$

where

$$A = \partial_{\eta_j}\left\{\frac{\langle\xi',\eta'\rangle + i|\xi'|\xi_n}{|\eta'|^2+\xi_n^2}\left[1 - \frac{\xi_n(\langle\xi',\eta'\rangle + i|\xi'|\xi_n)}{i|\xi'|(|\eta'|^2+\xi_n^2)}\right]\right\} = A_1 - A_2$$

and

$$A_1 = \frac{\xi_j(|\eta'|^2+\xi_n^2) - 2\eta_j(\langle\xi',\eta'\rangle + i|\xi'|\xi_n)}{(|\eta'|^2+\xi_n^2)^2}\left[1 - \frac{\xi_n(\langle\xi',\eta'\rangle + i|\xi'|\xi_n)}{i|\xi'|(|\eta'|^2+\xi_n^2)}\right]$$

$$A_2 = \frac{\langle\xi',\eta'\rangle + i|\xi'|\xi_n}{|\eta'|^2+\xi_n^2}\frac{\xi_n}{i|\xi'|}\frac{\xi_j(|\eta'|^2+\xi_n^2) - 2\eta_j(\langle\xi',\eta'\rangle + i|\xi'|\xi_n)}{(|\eta'|^2+\xi_n^2)^2}.$$

Through the computation,

$$\partial_{\xi_i}A_1|_\star = \frac{\delta_{ij}(1+\xi_n^2) - 2\xi_i\xi_j(1+i\xi_n)}{(1+\xi_n^2)^2}\left[1 - \frac{\xi_n(1+i\xi_n)}{i(1+\xi_n^2)}\right];$$

$$A_1|_\star = \frac{-i\xi_j}{(1-i\xi_n)^2(\xi_n+i)}; \quad B_1 := a_{n,m}\partial_{\xi_i}\left(\frac{A_1}{\xi_n - i|\xi'|}\right)|_\star = \frac{i a_{n,m}}{(\xi_n+i)^2(\xi_n-i)^2}\left(\delta_{ij} - \frac{i\xi_i\xi_j}{\xi_n+i}\right);$$

$$A_2|_\star = \frac{i\xi_j\xi_n}{(1-i\xi_n)^3}; \quad \partial_{\xi_i}A_2|_\star = \frac{\xi_n}{i(1+i\xi_n)(1-i\xi_n)^3}[\delta_{ij}(1-i\xi_n) - 2\xi_i\xi_j];$$

$$B_2 := a_{n,m}\partial_{\xi_i}\left(\frac{A_2}{\xi_n - i|\xi'|}\right)|_\star = \frac{a_{n,m}\xi_n}{(\xi_n+i)^2(\xi_n-i)^2}\left(\delta_{ij} - \frac{i\xi_i\xi_j}{\xi_n+i}\right);$$

$$D_{ij} = B_1 - B_2 = \frac{a_{n,m}}{(\xi_n-i)(\xi_n+i)^2}\left(-\delta_{ij} + \frac{i\xi_i\xi_j}{\xi_n+i}\right).$$

Using the fact that $\int_{|\xi'|=1}\xi_i\xi_j\sigma(\xi') = \pi\delta_{ij}$ and $\int_{|\xi'|=1}\sigma(\xi') = 2\pi$ where $|\xi'|=1$ is the unit circle, we have $i=j$ and

$$\begin{aligned}
\Omega_{2,\text{flat}}(f_1,f_2) &= \sum_{j=1}^{2}(-i)^3\partial_{x_j}f_1(x',0)\partial_{x_j}f_2(x',0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}D_{jj}d\xi_3\sigma(\xi')dx_1\wedge dx_2 \\
&= i\sum_{j=1}^{2}\partial_{x_j}f_1(x',0)\partial_{x_j}f_2(x',0)\int_{|\xi'|=1}\int_{\Gamma^+}a_{3,2}\frac{-1+\frac{i\xi_j^2}{\xi_n+i}}{(\xi_n-i)(\xi_n+i)^2}d\xi_3\sigma(\xi')dx_1\wedge dx_2 \\
&= i\sum_{j=1}^{2}\partial_{x_j}f_1(x',0)\partial_{x_j}f_2(x',0)\frac{a_{3,2}\pi i}{2}\left[-\frac{1}{2}\int_{|\xi'|=1}\xi_j^2\sigma(\xi') + \int_{|\xi'|=1}\sigma(\xi')\right] \\
&= -3\pi^2\sum_{j=1}^{2}\partial_{x_j}f_1(x',0)\partial_{x_j}f_2(x',0)dx_1\wedge dx_2 \\
&= -3\pi^2 d(f_1|_Y)\wedge\star d(f_2|_Y),
\end{aligned}$$

because X is flat and $a_{3,2}=4$.

b) Flat and f_1, f_2 Are Dependent of x_n Near the Boundary.

Since $n=3$ and $|\beta|\geq 1, |\delta|\geq 1$, so we have $|\delta|=|\beta|=1, |\alpha|=j=k=0$. By Lemma 6.1, then:

$$\Omega_{2,\mathrm{flat}}(f_1,f_2)=\sum_{|\beta|=1}\sum_{|\delta|=1}(-i)^3\partial_{x_n}^{\beta''}\partial_{x'}^{\beta'}f_1(x',0)\partial_{x'}^{\delta'}\partial_{x_n}^{\delta''}f_2(x',0)\times$$

$$\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\partial_{\xi'}^{\beta'}\partial_{\eta'}^{\delta'}\left\{\mathrm{trace}\left[\pi_{\xi_n}^+\partial_{\xi_n}^{\beta''}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}^{1+\delta''}\sigma_L(F)(\eta',\xi_n)\right]\right\}|_{\xi'=\eta'}d\xi_n\sigma(\xi')dx_1\wedge dx_2$$

$$=D_1+D_2+D_3+D_4,$$

where

$$D_1=\sum_{|\beta'|=1}\sum_{|\delta'|=1}(-i)^3\partial_{x'}^{\beta'}f_1(x',0)\partial_{x'}^{\delta'}f_2(x',0)\times$$

$$\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\partial_{\xi'}^{\beta'}\partial_{\eta'}^{\delta'}\left\{\mathrm{trace}\left[\pi_{\xi_n}^+\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right]\right\}|_{\xi'=\eta'}d\xi_n\sigma(\xi')dx_1\wedge dx_2;$$

$$D_2=\sum_{|\beta'|=1}(-i)^3\partial_{x'}^{\beta'}f_1(x',0)\partial_{x_n}f_2(x',0)\times$$

$$\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\partial_{\xi'}^{\beta'}\left\{\mathrm{trace}\left[\pi_{\xi_n}^+\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}^2\sigma_L(F)(\eta',\xi_n)\right]\right\}|_{\xi'=\eta'}d\xi_n\sigma(\xi')dx_1\wedge dx_2;$$

$$D_3=\sum_{|\delta'|=1}(-i)^3\partial_{x_n}f_1(x',0)\partial_{x'}^{\delta'}f_2(x',0)\times$$

$$\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\partial_{\eta'}^{\delta'}\left\{\mathrm{trace}\left[\pi_{\xi_n}^+\partial_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right]\right\}|_{\xi'=\eta'}d\xi_n\sigma(\xi')dx_1\wedge dx_2;$$

$$D_4=(-i)^3\partial_{x_n}f_1(x',0)\partial_{x_n}f_2(x',0)\times$$

$$\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\pi_{\xi_n}^+\partial_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}^2\sigma_L(F)(\xi',\xi_n)\right]d\xi_n\sigma(\xi')dx_1\wedge dx_2.$$

Observation: $D_1=-3\pi^2 d(f_1|_Y)\wedge\star d(f_2|_Y)$ by case a). In order to compute D_2, we must compute $\mathrm{trace}[\pi_{\xi_n}^+\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}^2\sigma_L(F)(\eta',\xi_n)]$. Instead of the way of Theorem 5.3, we compute $\pi_{\xi_n}^+\sigma_L(F)(\xi',\xi_n)$ firstly. Let $p(\xi',\xi_n)=\varepsilon(\xi)l(\xi)-l(\xi)\varepsilon(\xi)$ be a polynomial with matrices as coefficients where $\varepsilon(\xi)$ and $l(\xi)$ are the exterior and interior multiplications respectively, then

$$\sigma_L(F)=\frac{p(\xi',\xi_n)}{|\xi'|^2+\xi_n^2}$$

by Proposition 3.1 in [14]. By (2.1), we have:

$$\pi_{\xi_n}^+\left[\frac{p(\xi',\xi_n)}{|\xi'|^2+\xi_n^2}\right]=\frac{p(\xi',i|\xi'|)}{2i|\xi'|(\xi_n-i|\xi'|)};$$

$$\partial_{\xi_n}^2\sigma_L(F)(\eta',\xi_n)=\frac{\partial_{\xi_n}^2p(\eta',\xi_n)}{|\eta'|^2+\xi_n^2}-\frac{4\xi_n\partial_{\xi_n}p(\eta',\xi_n)}{(|\eta'|^2+\xi_n^2)^2}-\frac{2|\eta'|^2-6\xi_n^2}{(|\eta'|^2+\xi_n^2)^3}p(\eta',\xi_n).$$

Using $|$ instead of taking $\xi_n=i|\xi'|,\ \eta_n=\xi_n$, by Theorem 4.3 of [14] (n=odd case), then

$$T:=\mathrm{trace}[p(\xi)\times p(\eta)]=a_{n,m}\langle\xi,\eta\rangle^2+b_{n,m}|\xi|^2|\eta|^2$$

so,

$$T|=a_{n,m}\left[\langle\xi',\eta'\rangle+i|\xi'|\xi_n\right]^2;\ \ \partial_{\eta_n}T|=2ia_{n,m}|\xi'|\left[\langle\xi',\eta'\rangle+i|\xi'|\xi_n\right];\ \ \partial^2_{\eta_n}T|=-2a_{n,m}|\xi'|^2;$$

$$\mathrm{trace}\left[\pi^+_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial^2_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right]=\frac{1}{2i|\xi'|(\xi_n-i|\xi'|)}\times\left[\frac{1}{|\eta'|+\xi_n^2}\partial^2_{\eta_n}T|\right.$$
$$\left.-\frac{4\xi_n}{(|\eta'|^2+\xi_n^2)^2}\partial_{\eta_n}T|-\frac{2|\eta'|^2-6\xi_n^2}{(|\eta'|^2+\xi_n^2)^3}T|\right],\tag{7.1}$$

then compute the partial derivative ∂_{ξ_i} of (7.1) and take $\xi'=\eta'$ and $|\xi'|=1$, we have the result has form $\xi_if(\xi_n)$. Using $\int_{|\xi'|=1}\xi_i\sigma(\xi')=0$, so $D_2=0$. Similarly, we have $D_3=0$. In order to compute D_4, we'll compute

$$\mathrm{trace}[\pi^+_{\xi_n}\partial_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial^2_{\xi_n}\sigma_L(F)(\xi',\xi_n)]$$

. Since

$$\begin{aligned}
\pi^+_{\xi_n}\partial_{\xi_n}\sigma_L(F)(\xi',\xi_n)&=\pi^+_{\xi_n}\left[\frac{\partial_{\xi_n}p(\xi',\xi_n)}{|\xi'|^2+\xi_n^2}-\frac{2\xi_np(\xi',\xi_n)}{(|\xi'|^2+\xi_n^2)^2}\right];\\
\pi^+_{\xi_n}\left[\frac{\partial_{\xi_n}p(\xi',\xi_n)}{|\xi'|^2+\xi_n^2}\right]&=\frac{\partial_{\xi_n}p(\xi',\xi_n)|_{\xi_n=i|\xi'|}}{2i|\xi'|(\xi_n-i|\xi'|)};\\
\pi^+_{\xi_n}\left[\frac{2\xi_np(\xi',\xi_n)}{(|\xi'|^2+\xi_n^2)^2}\right]&=\frac{p(\xi',i|\xi'|)}{2i|\xi'|(\xi_n-i|\xi'|)^2}+\frac{\partial_{\xi_n}p(\xi',\xi_n)|_{\xi_n=i|\xi'|}}{2i|\xi'|(\xi_n-i|\xi'|)},
\end{aligned}$$

so,

$$\pi^+_{\xi_n}\partial_{\xi_n}\sigma_L(F)(\xi',\xi_n)=\frac{-p(\xi',i|\xi'|)}{2i|\xi'|(\xi_n-i|\xi'|)^2}.\tag{7.2}$$

Using (7.2), then

$$\mathrm{trace}[\pi^+_{\xi_n}\partial_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial^2_{\xi_n}\sigma_L(F)(\xi',\xi_n)]$$
$$=\mathrm{trace}\left\{\frac{-p(\xi',i|\xi'|)}{2i|\xi'|(\xi_n-i|\xi'|)^2}\times\left[\frac{\partial^2_{\xi_n}p(\xi',\xi_n)}{|\xi'|^2+\xi_n^2}-\frac{4\xi_n\partial_{\xi_n}p(\xi',\xi_n)}{(|\xi'|^2+\xi_n^2)^2}-\frac{2|\xi'|^2-6\xi_n^2}{(|\xi'|^2+\xi_n^2)^3}p(\xi',\xi_n)\right]\right\}$$
$$=ia_{n,m}\left[\frac{1}{(1+i\xi_n)^2(1+\xi_n^2)}+\frac{4i\xi_n}{(1+\xi_n^2)^2(1+i\xi_n)}+\frac{1-3\xi_n^2}{(1+\xi_n)^3}\right].$$

Integrate with respect to ξ_n, then

$$ia_{n,m}\int_{\Gamma+}\left[\frac{1}{(1+i\xi_n)^2(1+\xi_n^2)}+\frac{4i\xi_n}{(1+\xi_n^2)^2(1+i\xi_n)}+\frac{1-3\xi_n^2}{(1+\xi_n)^3}\right]d\xi_n=3\pi i.$$

So,

$$\begin{aligned} D_4 &= (-i)^3 \partial_{x_n} f_1(x',0) \partial_{x_n} f_2(x',0) \int_{|\xi'|=1} 3\pi i \sigma(\xi') dx_1 \wedge dx_2 \\ &= -6\pi^2 \partial_{x_n} f_1(x',0) \partial_{x_n} f_2(x',0) dx_1 \wedge dx_2 \\ &= -6\pi^2 \partial_{x_n} f_1(x',0) \partial_{x_n} f_2(x',0) \mathrm{Vol}_Y. \end{aligned}$$

Then we deduce the formula:

$$\Omega_{2,\mathrm{flat}}(f_1,f_2) = D_1 + D_4 = -3\pi^2 d(f_1|_Y) \wedge \star d(f_2|_Y) - 6\pi^2 \partial_{x_n} f_1(x',0) \partial_{x_n} f_2(x',0) \mathrm{Vol}_Y. \tag{7.3}$$

Remark: X has the product structure near the boundary, so $\partial_{x_n} f_i|_{x_n=0} \in C^\infty(Y)$.

c) Non-flat Case

Since $n=3$ and $r \leq -1, s \leq -1$, so $r = s = -1$, $|\alpha| = k = j = 0$ and $|\beta| = |\delta| = 1$, by (3.19) we have:

$$\Omega_2(f_1,f_2) = \sum_{|\beta|=1} \sum_{|\delta|=1} (-i)^3 \partial_x^\beta f_1(x',0) \partial_x^\delta f_2(x',0) \times$$

$$\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \mathrm{trace}\left[\pi_{\xi_n}^+ \partial_\xi^\beta \sigma_L(F)(\xi',\xi_n) \times \partial_{\xi_n} \partial_\xi^\delta \sigma_L(F)(\xi',\xi_n)\right] d\xi_n \sigma(\xi') dx_1 \wedge dx_2. \tag{7.4}$$

Observe: (7.4) is similar to case b) and the only difference is that $\sigma_L(F)$ is dependent of x. In the spirit of [10], we compute this form by the normal coordinate way.

In (7.4), there is no derivative ∂_{x_i} with respect to trace, so we take the normal coordinate and take boundary point $x = x_0$. Then $g^{ij}(x_0) = \delta_{ij}$ where $[g^{ij}]$ is the inverse matrix of metric matrix and this case is same to the case b). Whereas:

$$\begin{aligned} df_1(x',0) \wedge \star df_2(x',0)|_{x_0} &= \sum_{i,j=1}^{2} \partial_{x_i} f_1(x_0) \partial_{x_j} f_2(x_0) g^{ij}(x_0) \det{}^{\frac{1}{2}}[g_{ij}(x_0)] dx_1 \wedge dx_2 \\ &= \sum_{i=1}^{2} \partial_{x_i} f_1(x_0) \partial_{x_i} f_2(x_0) dx_1 \wedge dx_2 \end{aligned}$$

and $\mathrm{Vol}_Y|_{x_0} = dx_1 \wedge dx_2$, so (7.3) is correct in this case. By [3],

$$\Omega_2(f_1|_Y, f_2|_Y) = -8\pi d(f_1|_Y) \wedge \star d(f_2|_Y),$$

then we obtain:

Theorem 7.1 Suppose that (X,g) is a 3-dimensional compact oriented Riemannian manifold with boundary Y and has product metric near the boundary, then we have:

$$\begin{aligned} \Omega_2(f_1,f_2) &= -3\pi^2 d(f_1|_Y) \wedge \star d(f_2|Y) - 6\pi^2 \partial_{x_n} f_1(x',0) \partial_{x_n} f_2(x',0) \mathrm{Vol}_Y \\ &= \frac{3}{8}\pi \Omega_2(f_1|_Y, f_2|_Y) - 6\pi^2 \partial_{x_n} f_1(x',0) \partial_{x_n} f_2(x',0) \mathrm{Vol}_Y, \end{aligned}$$

$$\begin{aligned} &\widetilde{\mathrm{Wres}}(\pi^+ f_0 [\pi^+ F, \pi^+ f_1][\pi^+ F, \pi^+ f_2]) \\ &= -3\pi^2 \int_Y f_0|_Y \left[d(f_1|_Y) \wedge \star d(f_2|Y) + 2\partial_{x_n} f_1(x',0) \partial_{x_n} f_2(x',0) \mathrm{Vol}_Y \right]. \end{aligned} \tag{7.5}$$

The above formula is the generalization to manifolds with boundary of idea in [3] when $n = 3$. By [3], $\Omega_2(f_1|_Y, f_2|_Y)$ is conformally invariant. So although $\Omega_2(f_1, f_2)$ is not a conformal invariant, but we have:

Corollary 7.2

$$\Omega_2(f_1, f_2) + 6\pi^2 \partial_{x_n} f_1(x', 0)\partial_{x_n} f_2(x', 0)\mathrm{Vol}_Y$$

is a conformal invariant of (X, g).

Acknowledgement: The author would like to thank Professors Huitao Feng and Weiping Zhang for their helpful discussions. He also thanks the referee for his careful reading and helpful comments.

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